On Optimality of Quadrature Amplitude Modulation

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Abstract— **This paper demonstrates that for transmitting data over a strictly band-limited transmission channel of bandwidth B quadrature amplitude modulation at symbol rate $r_s = 2B$ is optimal with regard to information throughput. In accordance with the sampling theorem any band limited signal can be uniquely represented with $2B$ samples per second and reconstructed using reconstruction circuit which is in fact a generic QAM modulator. Band limited signals produced with any other modulation technique can thus also be obtained using a QAM modulator. This implies that QAM performs at least as good as any other modulation and is thus optimal.**

I. Introduction

The upper bound of the information rate r_i (number of information bits per second) over a noisy band-limited channel was first stated by C. E. Shannon [1]. One way to express this bound is:

$$r_i \leq \int_{f_l}^{f_u} C(f)\,df \tag{1}$$

where f_l and f_u are the lower and upper frequency bounds respectively, and $C(f)$ is the frequency dependant capacity of the channel (bits per second per Hertz). However, no known method for determining $C(f)$ of an arbitrary channel is known. The capacity can be determined only for a small number of special cases, e.g., additive Gaussian noise channel, where $C(f)$ can be expressed as:

$$C(f) = \log_2\left(1 + SNR(f)\right) \tag{2}$$

and $SNR(f)$ is the frequency dependant signal to noise ratio at the receiver input. Even for this simplified case it can be hard or even impossible to determine the signal to noise ratio at the receiver input when the channel is non-linear.

Thus, in general it is not known how close to the Shannon bound different modulation techniques are. In recent years this led to many attempts of discovering a new modulation technique which could improve on all currently known methods in terms of spectral efficiency, e.g., FQPSK (Feher-Patented Quadrature Phase-Shift Keying) [2], VPSK (Variable Phase Shift Keying) [3], VMSK (Very Minimum Shift Keying) [4], [5] and many other ultra-narrow band modulations [6], [7], which were supposed to achieve a spectral efficiency a whole order of magnitude greater than all previously known methods.

In my previous paper [8] I already showed that no ultra-narrow band modulation can exceed the spectral efficiency of QAM. In this paper, I wish to further show that QAM at symbol rate $r_s = 2B$ is optimal. To be more specific, if there exists any modulation technique that approaches the Shannon bound for the given channel than the same can also be achieved using QAM at symbol rate $r_s = 2B$.

The paper is organized as follows. In section II we first describe the model of band-limited transmission and then define the equivalency of modulation. In section III we review sampling and reconstruction of band limited signals. We use the findings of section III in section IV to show that the QAM is equivalent to any band-limited modulation, thus it is also equivalent to the optimal modulation, and is itself optimal. In section V we make brief discursion of the result obtained.

II. Band-limited transmission

We will investigate the band-limited transmission system shown in Fig. 1.

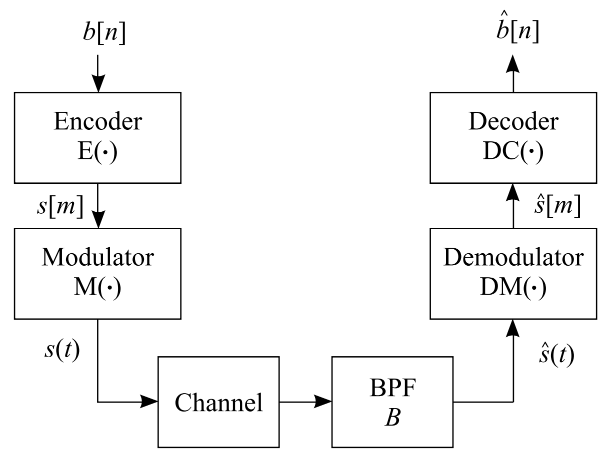

Fig. 1. Band-limited transmission system

The input bit stream $b[n]$ at the bit rate r_b is encoded to the stream of real valued symbols $s[m]$ at any chosen symbol rate r_s:

$$s[m] = \mathrm{E}(s[n]) \tag{3}$$

where $\mathrm{E}(\cdot)$ denotes the transformation performed by the encoder. Symbols $s[m]$ are then led to the input of the modulator. The transmitted signal $s(t)$ at the output of the modulator can be expressed as:

$$s(t) = \mathrm{M}(s[m]) = \mathrm{M}(\mathrm{E}(b[n])) \tag{4}$$

where $\mathrm{M}(\cdot)$ denotes the transformation performed by the modulator. We assume that the transmission is band-limited, thus the signal $s(t)$ at the output of the modulator is band-limited to the frequency band of bandwidth B.

It is important to note that within the scope of this paper, the term coding is used for mapping one symbol stream to another symbol stream and the term modulation is used for mapping a symbol stream to a continuous signal. Other definitions of coding and modulation can be found in other sources.

When the channel is non-linear it produces out-of-band frequency components. These components can also carry information. As we assume that the transmission is band-limited, the information transmitted out-of-band should be irrelevant for the receiver. To ensure that no information is carried out of the transmission band an ideal band-pass filter (BPF) of bandwidth B is included at the output of the channel in front of the receiver. The band-limited signal $\hat{s}(t)$ at the receiver input is first demodulated to the received symbol stream $\hat{s}[m]$:

$$\hat{s}[m] = \mathrm{DM}(\hat{s}(t)) \tag{5}$$

and then decoded to yield the received bit stream $\hat{b}[n]$:

$$\hat{b}[n] = \mathrm{DC}(\hat{s}[m]) = \mathrm{DC}(\mathrm{DM}(\hat{s}(t))) \tag{6}$$

where $\mathrm{DM}(\cdot)$ and $\mathrm{DC}(\cdot)$ denote transformations made by demodulator and decoder respectively.

A. *Equivalent modulation*

We say that the encoder-modulator pair (E',M') is equivalent to the encoder-modulator pair (E,M) when and only when they produce the same output for the same input:

$$\mathrm{M'}(\mathrm{E'}(b[n])) = \mathrm{M}(\mathrm{E}(b[n])) \tag{7}$$

for each input bit stream $b[n]$.

Similarly we say that the demodulator-decoder pair (DM',DC') is equivalent to the demodulator-decoder pair (DM,DC) when and only when:

$$\mathrm{DC'}(\mathrm{DM'}(\hat{s}(t))) = \mathrm{DC}(\mathrm{DM}(\hat{s}(t))) \tag{8}$$

for each signal $\hat{s}(t)$.

As we are concerned with modulation only, not with coding, we can define equivalency of modulation as follows:

- The modulation M' is equivalent to the modulation M when for an arbitrary encoding E there exists an encoding E' such that the pairs (E',M') and (E,M) are equivalent,
- the demodulation DM' is equivalent to the demodulation DM when for an arbitrary decoding DC exists a decoding DC' such that (DM',DC') is equivalent to (DM,DC), and finally
- two modulation methods are equivalent when both modulation and demodulation are equivalent.

III. Sampling and reconstruction of band-limited signals

Let signal $s(t)$ be a strictly band-limited signal of bandwidth B. It can be sampled with the band-pass sampler (BPS) shown in Fig. 2.

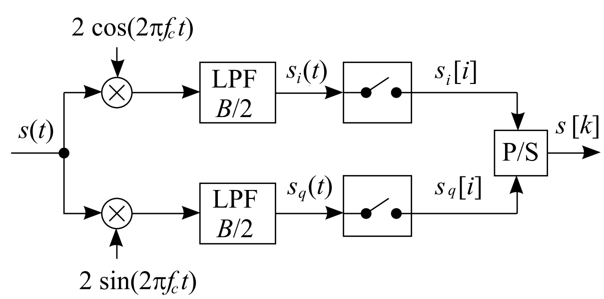

Fig. 2. Sampling of band-limited signal

The signal is first multiplied by two orthogonal harmonic signals 2 $\cos(2\pi f_c t)$ and 2 $\sin(2\pi f_c t)$, where f_c is the central frequency of the transmission band:

$$f_c = \frac{f_u + f_l}{2} \tag{9}$$

With low-pass filters (LPFs) of bandwidth $B/2$ all components around the double central frequency are removed. Signals $s_i(t)$ and $s_q(t)$ are sampled at time intervals $T_s = 1/B$. The obtained symbol streams $x_i[i]$ and $x_q[i]$ are then combined on a parallel to serial converter into a single symbol stream $s[k]$ at symbol rate $r_s = 2B$.

The signal $s(t)$ can be perfectly reconstructed from the symbol stream $s[k]$ with the band-pass reconstructor (BPR) shown schematically in Fig. 3.

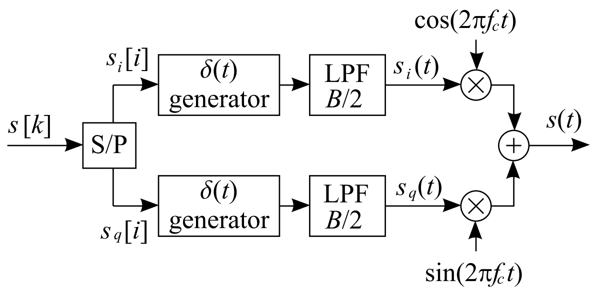

Fig. 3. Reconstruction of band-limited signal

With the serial to parallel converter, the symbol stream $s[k]$ is converted back into streams $s_i[i]$ and $s_q[i]$. Since the bandwidth of signals $s_i(t)$ and $s_q(t)$ was $B/2$, and the sampling frequency was B, i.e., double the bandwidth of the signals, we can, in accordance with the sampling theorem [9], perfectly reconstruct signals $s_i(t)$ and $s_q(t)$ from their samples $s_i[i]$ and $x_q[i]$ with the use of two ideal LPFs with bandwidth $B/2$.

The signals at the output of LPFs are multiplied by $\cos(2\pi f_c t)$ and $\sin(2\pi f_c t)$ and then summed up to yield original signal $s(t)$.

The proof of the above can be found in almost any elementary text on communications theory (see for example, [10],[11]). We would only like to point out that the BPR on Fig. 3 is in fact a generic quadrature amplitude modulator (QAM) which maps the symbol stream $s[k]$ to the signal $s(t)$ and BPS on Fig. 2 is QAM demodulator which maps the signal $s(t)$ back to the symbol stream $s[k]$.

IV. Optimality of QAM

To show that QAM is optimal we first assume that the modulation method used in Fig. 1 is optimal for the given channel, i.e., that the encoding-modulation (E,M) and demodulation-decoding (DM,DC) pairs on Fig. 1 approaches the Shannon bound as closely as possible.

Now we sample and reconstruct the signals $s(t)$ at the transmitter and $\hat{s}(t)$ at the receiver using two band-pas samplers (BPSs) and two band-pass reconstructors (BPRs) as shown in Fig. 4. Since both signals are band-limited, the reconstruction is perfect.

Let us now examine the system in Fig. 4 in more detail. At the transmitter side, we convert the output stream $b[n]$ to stream $s[k]$ with the help of the encoder,

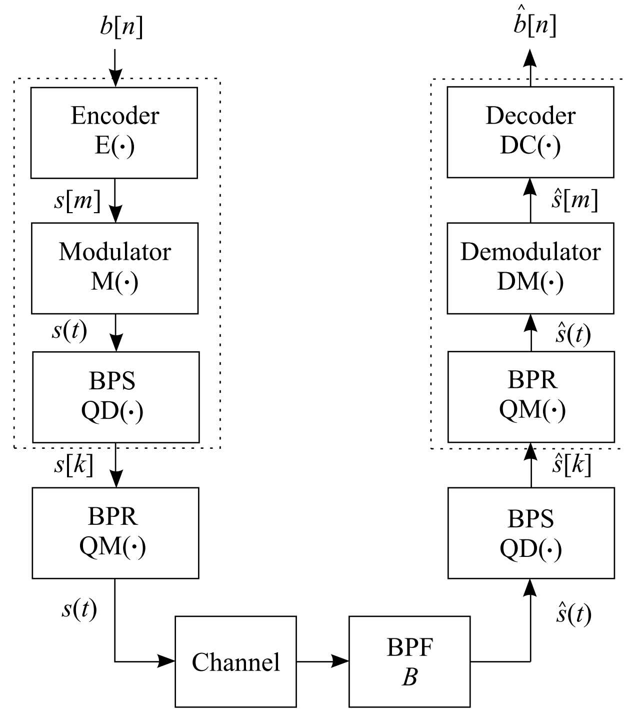

Fig. 4. Additional samplings and reconstructions

modulator and BPS:

$$s[k] = \mathrm{QD}(\mathrm{M}(\mathrm{E}(b[n]))) \tag{10}$$

where QD(·) denotes transformation performed by band-pass sampling. The serial connection of the encoder, modulator and BPS thus represent a new encoder that performs the transformation:

$$\mathrm{E'}(b[n]) = \mathrm{QD}(\mathrm{M}(\mathrm{E}(b[n]))) \tag{11}$$

The BPR is connected at the output of this new encoder. As we already mentioned the BPR is in fact QAM modulator (QM), thus we have a new encoder-modulator pair (E',QM).

Similarly, we can conclude that BPS at the input of the receiver represents a QAM demodulator (QD), while the combination of BPR, demodulator and decoder represents a new decoder that performs the transformation:

$$\mathrm{DC'}(\hat{s}[k]) = \mathrm{DC}(\mathrm{DM}(\mathrm{QM}(\hat{s}[k]))) \tag{12}$$

which converts the symbol stream $\hat{s}[k]$ to the bit stream $\hat{b}[n]$. We have a new pair (QD,DC'). The situation is shown in Fig. 5.

Since the reconstruction of signal $s(t)$ was perfect, we can conclude:

$$s(t) = \mathrm{QM}(\mathrm{E'}(b[n])) = \mathrm{M}(\mathrm{E}(b[n])) \tag{13}$$

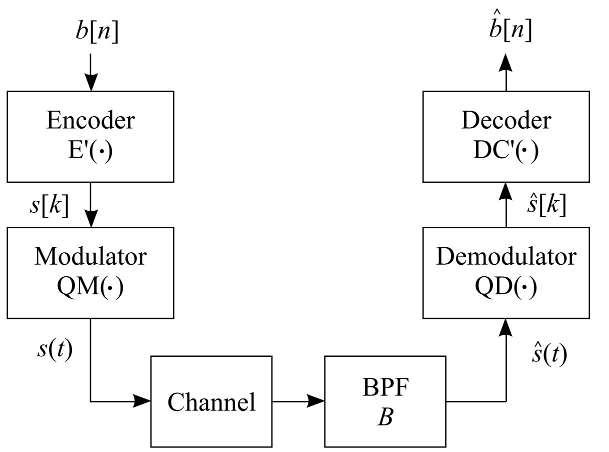

Fig. 5. Equivalent system based on QAM

and the pair (E',QM) is equivalent to the pair (E,M). Similarly we can conclude from

$$\hat{b}[n] = \text{DC'}(\text{QD}(\hat{s}(t))) = \text{DC}(\text{DM}(\hat{s}(t))) \qquad (14)$$

that the pairs (QD,DC') and (DM,DC) are equivalent.

From (11) and (13) we see, that E' which make (E',QM) equivalent to (E,M) exists. Thus, in accordance with the definition in section II-A, QM is equivalent to M. Similarly we can conclude from (12) and (14) that QD is equivalent to DM.

At the beginning of this section we assumed that modulation M and demodulation DM are optimal. We showed that QAM modulation (QM) and QAM demodulation (QD) are equivalent to M and DM respectively, thus they are also optimal.

From above we can conclude that QAM at symbol rate $r_s = 2B$ is an optimal modulation method. The optimal coding and decoding methods E' and DC' belonging to QAM are different from what they would be if we had chosen some other modulation. In general the optimal coding method for an arbitrary channel is not known. However, whenever the optimal coding method belonging to some other modulation is known, according to equations (11) and (12), the optimal method belonging to QAM is known too.

V. Conclusions

We have shown that the QAM at symbol rate $r_s = 2B$ is an optimal modulation method. The space of signals provided by QAM includes the output of all other strictly band limited modulation methods, thus by introducing a QM/QD pair any other modulation method can be implemented in the encoding. We can only adapt to the characteristic of the given channel through a suitable choice of coding, which indicates that the research should be focused on finding optimal coding methods instead of being focused on discovering new modulation techniques.

Actually, what we showed is, that any strictly band limited modulation can be implemented using QAM modulator at a symbol rate $r_s = 2B$. This is a consequence of the fact that any band-limited signal can be uniquely represented with $2B$ samples per second. Although, at least theoretically, any modulation can be implemented with QAM, this may not be the most practical solution.

Since the coding methods currently used are already rapidly approaching the upper bound for most of the channels of interest, we cannot expect significant progress in this area. The point of further research therefore lies not in discovering new, more efficient methods, but rather in discovering methods which can more readily be implemented technologically, more robust sub-optimal methods, or exploiting space diversity in multi-path propagation.

Acknowledgment

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