Coding gain and lattice codes with small number of symbols

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ABSTRACT

Constellation diagrams are an important factor in channel coding, representing the physical properties of symbols when they are mapped to electrical impulses. An abstract phenomenon is thus converted into an electrical impulse. Transmission efficiency can be improved by increasing the number of orthogonal carriers, by shaping the whole constellation diagram and by correct positioning of symbols (points) in the diagram. Most common coding schemes do not employ optimum constellation diagram point distributions, so it is obvious that the common method of using equidistant independent direction symbol packing is not desirable. Symbols with added noise are geometrically modelled as spheres, so the coding problem is reduced to the problem of sphere packing of n-dimensional hyperspheres. This problem can be solved with the use of lattices, which can be quite efficiently represented with the algebraic groups. Because lattices are used, this is called "lattice coding". Simple mathematical relations only apply in special cases where there are a large number of symbols; much more effort must be used in determining coding with a small number of symbols. Searching for coding schemes with minimal energy for a given symbol number yields actual coding gain or efficiency. Small number of symbols seem to give very specific results.

Keywords: multi-dimensional coding, small number of symbols, groups, lattices, sphere packing

1. INTRODUCTION

The theoretical upper limit of coding was defined by Shannon a long time ago, but searching for optimal coding has continued to the present day.

Transmission efficiency can be affected by the number of orthogonal carriers, by shaping the constellation diagram and by correct positioning of symbols (points) in the diagram. Proper distribution of points in the constellation diagram can improve the average power of coding and a proper shape of the diagram can improve the peak-to-average power ratio (PAR).

Most common coding schemes do not employ optimum point distribution. The basic shape is an equidistant symbol set in the orthogonal directions. In two-dimensional spaces this shape corresponds to the constellation diagram of Quadrature Amplitude Modulation (QAM). In multi-dimensional spaces the described shape can be generalized.

Probability of error is related to the distance to the nearest symbol, so it can be noted that this is not an optimal shape. If an equidistant constellation diagram in two-dimensional space is required, a distribution with 6 neighbours for each symbol results. This structure can be repeated infinitely and resembles a honeycomb. Such infinite structures are called lattices and can be mathematically described as groups.

Shannon postulated that the symbol arrangement is not important in spaces with an infinite number of dimensions. But, since we only have a finite number of dimensions and efficient decoding is required, the theoretical capacity can never be achieved:

$$C = \tfrac{1}{2}\log_2\left(1+\tfrac{S}{N}\right). \quad (1)$$

2. CODING GAIN AND SHAPING GAIN

The simplest multi-level one-dimensional coding is shaped as an equidistantly distributed set of symbols. Multi-dimensional constellation diagrams can be gotten from generalizing the single-dimensional example.

The described constellation diagram type is simple to implement both in the transmitter and the receiver and is frequently used due to its good performance. We are trying to find coding schemes that are more efficient, compared to the basic model. When searching for new constellation diagrams, coding gain is first considered. Equal SNR is desirable, determined by the shortest distance between two symbols. Coding efficiency is also determined by Peak-to-Average power Ratio (PAR), scalability and complexity of practical coding.

Coding comparisons can be divided into two parts:

- coding gain, γ_c
- shaping gain, γ_s.

These two parameters can be looked at separately, but both must be used to make a practical comparison. For them to be independent of each other, the probability of all symbols must be equal. Shaping gain is not so important, because we already know the optimal shape (spherical) and its shaping gain (1.53 dB). Distributing symbols can be much more difficult, because an optimal constellation is not known.

Coding gain is increased with various constellations, sometimes optimised with the aid of the knowledge of channel properties (eg. Gauss or Rayleigh channel [9]). Usually a pattern is used, which is then infinitely repeated. This constitutes a lattice and is a geometrical description of groups.

Coding (γ_c) and shaping (γ_s) gain definitions [5]:

$$\gamma_C(\Lambda) = \frac{d^2_{\min}(\Lambda)}{V(\Lambda)^{2/n}}, \quad (2)$$

$$\gamma_S(\Lambda) = \frac{V(R)^{2/n}}{6 \cdot P(R)}. \quad (3)$$

Λ represents the constellation diagram point distribution, R is the space contained in the constellation diagram and $V(R)$ its volume. $P(R)$ is the average energy or coding power and $d_{\min}$ is the shortest distance between points in the constellation diagram.

3. OPTIMAL PACKING, LATTICES AND CODING SCHEMES

Symbols in the constellation diagram can be chosen at will, but are usually a part of a pattern for practical reasons (coding and decoding). This means all distances between neighbouring points are the same. This distribution is usually based on a pattern that can be infinitely repeated. Such patterns are called "lattices" and this form of coding is known as "lattice coding".

A lattice can be represented geometrically (figure 1). It can also be represented as a group, although this is less easily understood. A group is represented as a set and an operation. Previous example yields a set of points (two-dimensional place vectors) and operation of addition. Addition of any place vectors yields a new point, still a member of the set.

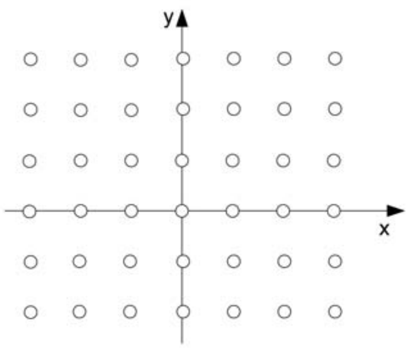

Figure 1: Z^2 lattice

Definition of lattice Λ is [3]:

$$\lambda_1 \cdot v_1 + \lambda_2 \cdot v_2 + \ldots + \lambda_m \cdot v_m; \;\; \lambda_1, \lambda_2, \ldots, \lambda_m \in Z \quad (4)$$

The base of the lattice is formed with a set of vectors $\{v_1, v_2, \ldots, v_m\}$, where m represents the lattice dimension or rank. If $m=n$ (n — space dimension), the lattice is full-ranked.

A lattice can also be described with the aid of a generator matrix consisting of base vectors $\{v_1, v_2, \ldots, v_m\}$:

$$M = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix} = \begin{bmatrix} v_{11} \cdots v_{1n} \\ \vdots \\ v_{m1} \cdots v_{mn} \end{bmatrix}. \quad (5)$$

The above m represents the dimension of the lattice and n the dimension of the space (v_{xx} are coordinates of vectors in single dimensions). If two lattices only differ in scaling factor and rotation, they are considered equivalent.

The space around points can be divided so that every point occupies the same amount of space. Coding with hard boundaries means that the boundaries of elementary spaces correspond to boundaries between symbols. Space claimed by one point is called a Voronoi cell. A point of the lattice Λ in space R^n is represented by s_i and a Voronoi cell is defined $\upsilon(s_i)$ [3].

Many times, the volume of this cell needs to be known. In full-ranked lattices the volume is equal to the determinant of the generator matrix. Volume is independent of lattice base, but the shape of an elementary parallelotope is not. Volume of the basic space segment of the lattice Λ ($|\det(M)|$) can also be represented as $d(\Lambda)$ or $vol(\Lambda)$.

Lattices that are not full-ranked ($m<n$, n – space dimension) do not permit the calculation of volume with the aid of the generator matrix determinant. Volume can be obtained with the use of Gram-Schmidt orthogonalization yielding a Gram matrix (A – Gram matrix, M – generator matrix of lattice Λ):

$$A = M \cdot M^T. \quad (6)$$

Universal validity [3] can be proven for the equation:

$$d(\Lambda) = \sqrt{\det(A)}. \quad (7)$$

Also for scaled cases (scaling constant is c):

$$d(c \cdot \Lambda) = c^n \cdot d(\Lambda). \quad (8)$$

3. 1 The sphere packing problem

The optimal packing problem has long been known in mathematics, described by packing identical multi-dimensional spheres. The problem is solved when the maximum number of spheres is packed into given space. Spheres cannot fill the entire space, so the packing density is represented by Δ.

One of the possible solutions to the problem is so-called “lattice packing”. A full-ranked lattice is taken as a basis and identical spheres are placed in the lattice with their centres corresponding to points in the lattice. The distance between neighbouring lattice points is d_{min}, therefore spheres with radius $\rho = d_{\min}/2$ are used. The spheres touch but do not overlap. A sphere with radius ρ is also the largest sphere that can fitt inside a Voronoi cell. Packing density (Δ) is the quotient of sphere volume and Voronoi cell volume.

Another important parameter is the “kissing number” which tells us how many times a sphere touches neighbouring spheres. Generally speaking, the kissing number could be different for each sphere, but with lattices, the kissing number is constant for all spheres. [7] gives us a table of densest lattice solutions.

3. 2 Coding and sphere packing problem

Symbols and electrical signals are linked by a constellation diagram that assigns a point (or a vector) inside a multi-dimensional space to every symbol. The Euclidian norm of the vector x represents the energy of the transmitted signal.

During transmission, noise η is added to the signal x. The receiver must thus extract a proper value out of the sum $y = x + \eta$. If we use coding with a set $\{s_1, s_2, \ldots s_M\}$, then the Voronoi cell around each point is represented with $\upsilon(s_k)$. If a signal s_k was transmitted, the receiver can decode it correctly only if the signal value at the receiver input is within the Voronoi cell $\upsilon(s_k)$.

The probability for this to happen is:

$$P = \frac{1}{(\sigma \cdot \sqrt{2\pi})^n} \int_{\upsilon(s_k)} e^{-\frac{\|x\|^2}{2\sigma^2}} dx \,. \tag{9}$$

If a constellation is constructed on the basis of a lattice, we call it a “lattice constellation”. The error (e) probability is then:

$$P(e) = 1 - P \,. \tag{10}$$

This equation is only valid when the constellation consists of an infinite number of points ($M \rightarrow \infty$), obviously not a practical case. Error is introduced at the lattice boundaries, because there the Voronoi cells sometimes have unlimited dimensions. This equation is nevertheless useful for coding with a large set of symbols, because the boundary effects can then be disregarded. Coding with an n-dimensional lattice having a determinant $d(\Lambda)$ using a Gaussian channel with standard deviation σ gives a signal-to-noise ratio [9]:

$$SNR_{dB} = 10 \cdot \log \frac{[d(\Lambda)]^{2/n}}{4 \cdot \sigma^2} \,. \tag{11}$$

Lattice constellations make it very difficult to compute error probability $P(e)$. The integration of spaces that are difficult to describe is required. Approximations are thus used, usable only for fixed S/N ratios. These equations are based on the premise that every symbol has the same probability and that the number of symbols is large enough to offset boundary effects.

With a large SNR and equidistant points the probability of error [3] is:

$$P(e) \approx \frac{\tau}{2} \cdot erfc\left(\frac{\rho}{\sigma \cdot \sqrt{2}}\right) . \tag{12}$$

The above equation requires the knowledge of the kissing number τ and the sphere radius ρ. There is also a simpler way to describe the SNR if the set has a large enough number of symbols M and the maximum coding power C^2 is known [3]:

$$SNR'_{dB} = 10 \cdot \log \frac{C^2/n}{2 \cdot \sigma^2 \cdot M^{2/n}} \,. \tag{13}$$

A new way to express error probability can thus be used [3]:

$$P(e) \approx \frac{\tau}{2} \cdot erfc\left(\sqrt{\Delta^{2/n} \cdot n \cdot SNR'_{dB}}\right) . \tag{14}$$

Coding gain is defined with regard to distribution Z^n. For a high SNR it can be shown [3]:

$$\gamma_C(\Lambda) = \frac{d^2_{\min}}{\sqrt[n/2]{d(\Lambda)}} = 4 \cdot \sqrt[n/2]{\delta} \,. \tag{15}$$

Table 1 shows coding gains for few different lattice codings.

Λ	$\gamma_C(\Lambda)$ /dB
A_2	0.62
D_4	1.50
E_6	2.21
E_8	3.01
K_{12}	3.63
Λ_{16}	4.51
Λ_{24}	6.02

Table 1: Coding gain

3. 3 Decoding

Lattice coding is theoretically very efficient. De Buda [8] has shown that they approach Shannon's predictions.

The major drawback, however, is the fact that efficient decoding is difficult, thus, for the time being, precluding wider acceptance. Several universal decoding procedures are known, employing closest point searching, but these take too long to execute. Also some efficient special case algorithms are known, for e.g. A_n, D_n and Leech Λ_{24} lattices. These algorithms are optimized to use the special properties of these lattices.

Another type of decoding is also known, employing Viterbi's algorithm. This approach can be used with finite lattices, where the lattice is presented as a trellis diagram.

3.4. Frequently used lattice properties

Table 2 gives the most important lattice properties (determinant, sphere radius and kissing number) of most frequently used lattices (n – space dimension).

Λ	$d(\Lambda)$	ρ	τ
Z^n	1	$1/2$	$2n$
A_n (n≥2)	$\sqrt{n+1}$	$1/\sqrt{2}$	$n(n+1)$
D_n (n≥3)	2	$1/\sqrt{2}$	$2n(n-1)$
E_6	$\sqrt{3}$	$1/\sqrt{2}$	72
E_7	16	1	126
K_{12}	27	1	756
Λ_{16}	16	1	4320
Λ_{24}	1	1	196560

Table 2: Lattice properties

4. CODING GAINS WITH SMALL NUMBER OF SYMBOLS

The previously presented coding equations are limited to coding schemes with a large number of symbols. When a small number of symbols are used, the situation gets more complicated.

A simple computer simulation was used to determine coding gain dependence on the number of symbols. This paper presents results for coding schemes based on lattices A_2 and A_3.

First, coding gains for a large number of symbols are calculated. This is done with the aid of equation (15), $d(\Lambda)$ and $d_{\min}$ can be found in table 2:

$$d_{\min} = 2 \cdot \rho$$

The result:

$$\gamma_C(A_2) = 1.15 = 0.62dB$$

$$\gamma_C(A_3) = 1.26 = 1.00dB$$

These values will now be compared to actual coding gain for chosen number of symbols. The simulation algorithm is created in such a way that it creates a coding scheme with minimal power.

Coding gain of a coding scheme tells us how much power is saved compared to the use of Z^n lattice. Reference and comparison schemes must have an identical number of symbols.

Figures 1 and 2 show coding gain of coding schemes based on A_2 and A_3 lattices with different number of symbols.

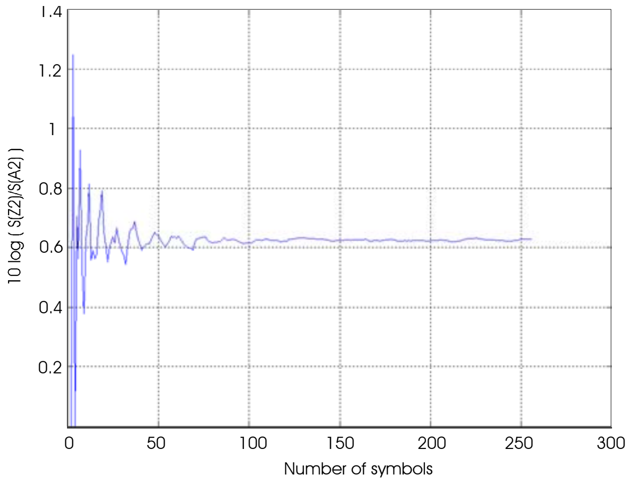

Figure 2: A_2 lattice coding gain dependence on the number of symbols

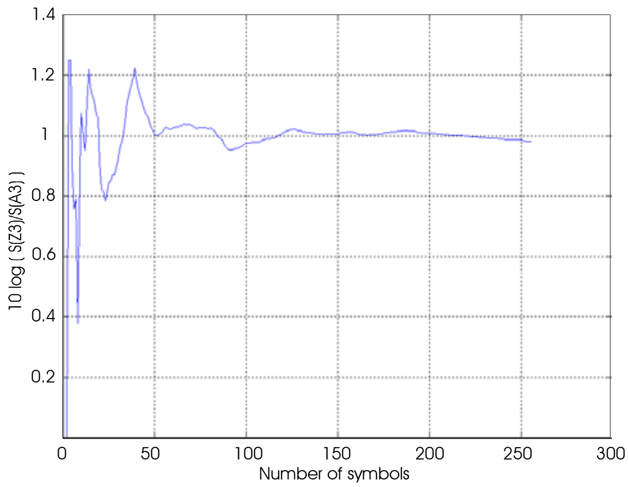

Figure 3: A_3 lattice coding gain dependence on the number of symbols

We can see that the coding gain changes a lot with small number of symbols, but becomes progressively more constant as the number of symbols increases, approaching theoretical estimates.

Figures 4 and 5 show detailed results for small number of symbols (<50). Changing coding gains can be attributed to shaping gain, because there are few possibilities to shape the coding scheme.

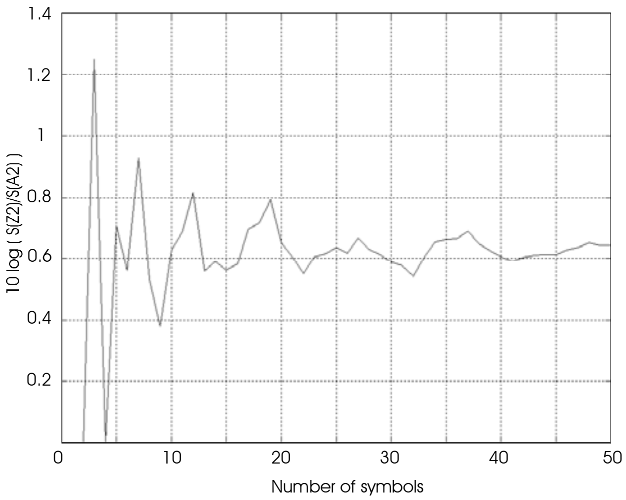

Figure 4: Coding gain of A_2 lattice with small number of symbols

Limited shaping influences both decreases and increases of coding gain compared to the theoretical value. If a certain number of symbols is used, sometimes lattice A_n is optimal and sometimes Z^n.

These effects are especially visible in figure 4, where A_2 coding results are shown. High coding gain for three symbols is a consequence of equal minimal distances between the three symbols. When four symbols are used, coding gain is 0dB, because the optimal form is the same as the Z^2 form. There is also an obvious gain for seven symbols, which are optimal for A_2 lattice – the centre symbol is surrounded with six symbols, each having same Euclidian distances from it.

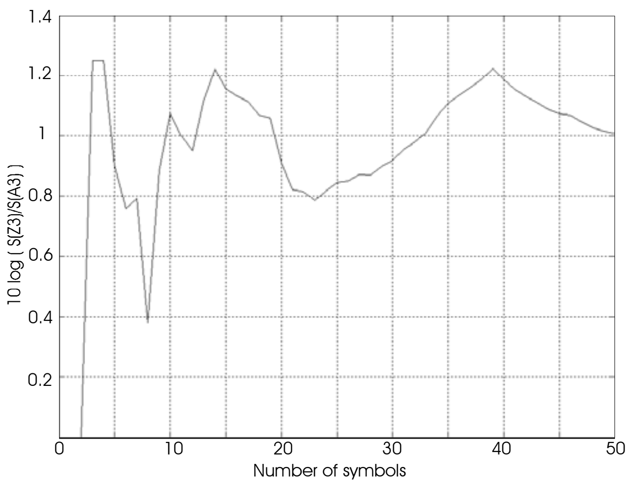

Figure 5: Coding gain of A_3 lattice with small number of symbols

5. CONCLUSION

The literature is full of ways to improve coding to increase bit rates and decrease power. One of them is lattice coding, as presented in this paper. Coding can be separated into shaping and distribution of symbols, which can sometimes make our job easier. The theoretical best coding gain that can be achieved with constellation diagram shaping is known, as well as the procedures or shapes that give the best gain with easiest practical application. Here we can mention Voronoi and cross constellation.

Constellation points distribution is a more difficult problem. The probability of errors is most influenced by neighbouring symbols, which is why we are searching for the densest distribution possible. One possible way to find this distribution is spherical packing, but optimal solutions are only known for one- and two-dimensional spaces. Multi-dimensional spaces can only be solved by approximation. All solutions have the interesting and useful mathematical property that the packing pattern repeats itself indefinitely, which is why we can describe them as lattices. We can derive relatively simple equations for coding gain, error probability, SNR, etc. The problem is that they are limited to specific conditions. One possible problem is small numbers of symbols, which has been explored in this paper. Computer simulation has shown what the problems are. We can not solve them, but sometimes we can work around them.

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